LQG cheap control subject to packet loss and SNR limitations

A. Chiuso, N. Laurenti, L. Schenato, A. Zanella

Abstract— **In this paper we consider the problem of controlling unstable stochastic linear systems in the presence of a communication channel between the sensors and the actuators. We propose an LQG architecture that separates the problem of designing suitable regulators for controlling the plant, referred to as Plant Encoder/Decoders, from the problem of designing encoder/decoder for the communication channel. We provide a mathematical model that takes into account the most important features of today's wireless communication protocols such as quantization errors, limited channel capacity, decoding delay and packet loss, while still being amenable to analytic treatment. We then restrict our discussion to a special class of linear plant encoder/decoders and to a channel with signal-to-noise (SNR) limitations and packet loss only, and we derive stability conditions and optimal parameters for the controller design in the cheap-control setting. Through this analysis we are able to recover several results available in the literature that treated packet loss and quantization error separately.**

I. Introduction

Traditionally control theory and communication theory have been developed independently and have reached considerable success in developing fundamental tools for designing information technology systems. On one side, the major objective of control theory was to develop tools to stabilize unstable plants and optimize some performance metrics in closed loop under the assumption that the communication channel between sensors and controller and between controller and plant were ideal, i.e. without distortion, packet loss or delay. On the other side, the major objective of communication theory was to develop tools to transmit information from a stable source to a receiver though a possibly noisy communication channel where the communication protocols had no feedback on the source. One of the reasons for the success of these theories was that, in many control applications, the effects of the communication channel impairments was negligible as compared to the effects of noise and uncertainty in the plants, while in many communication applications the time dynamics of the source statistics was slow as compared to the communication speed of the protocols, so that the source could be safely assumed to be stationary. With the advent of wireless communication, the Internet and the need for high performance control systems, this sharp separation between control and communication has begun to be questioned and a growing body of literature has appeared from both the communication and the control community, that tries to analyze the interaction between control and communication.

One line of research has addressed the problem of stabilization of an unstable plant through a rate-limited erasure channel where no performance index is considered besides stability [11], [21], [9]. Another line of research has applied information theoretic tools finding relationships between feedback performance and channel capacity [10], and showing that Shannon's capacity is not sufficient to characterize a communication channel from a control perspective [14]. Differently, other researchers have tried to tackle the channel limitations by using analog models in order to avoid the difficulties associated with explicit design of encoder/decoder for digital transmission, and to optimize some performance metrics among all possible stabilizing controllers [5], [13], [18]. Along these lines, other groups have modeled the limited capacity of the channel through a constraint on the maximum signal-to-noise (SNR) ratio and found fundamental limits that depend on the unstable eigenvalues of the plant [2], [17], [3]. Finally, another well explored approach is the analysis of control systems subject to random packet loss [19], [6], [7], [16] under an LQG framework.

The above cited works are just a partial overview of the literature on control systems subject to communication channel limits that is by no means complete. Indeed, the current trend is to include multiple channel limitations into the model, such as packet loss and quantization [22], [8], which however results in complex optimization problems.

The objective of this work is twofold. The first objective is to provide a more realistic model of a communication channel that, while still being mathematically amenable to analysis, includes packet loss, delay, SNR-limitations and quantization distortion. The second objective is to propose an LQG approach for the design of the control blocks in order to include performance metrics besides stability of the closed loop system. In fact, from a practical standpoint, stability is not sufficient and additional performance criteria need to be satisfied, such as in the LQG framework.

Although we introduce a very general architecture for networked control systems, in this work we limit our analysis and design to a simplified channel model that only includes packet loss and SNR limitations, and to a special class of linear controllers under the standard LQG cheap-control setting. Nonetheless, we recover several results available in the literature and find a stability condition that depends on both the packet loss probability and the SNR of the channel. This work extends to a multivariable setting some previous results which were limited to scalar plants [12].

This work is supported by the European Community's Seventh Framework Programme [FP7/2007-2013] under grant agreement n. 257462 HYCON2 Network of excellence

All authors are in the Department of Information Engineering, University of Padova, Via Gradenigo 6/b, 35131 Padova, Italy {chiuso|nl|schenato|zanella}@dei.unipd.it

II. Problem formulation

We consider the problem of stabilizing a possibly unstable system across a communication channel. The plant is modeled as a discrete-time linear time-invariant dynamical system subject to additive measurement and process noise. More specifically:

$$\begin{aligned} x_{t+1} &= Ax_t + Bu_t + w_t && (1) \\ y_t &= Cx_t + v_t && (2) \end{aligned}$$

where $x \in \mathbb{R}^n, u \in \mathbb{R}^p, y \in \mathbb{R}^m$, $v_t \sim \mathcal{N}(0,R)$, $w_t \sim \mathcal{N}(0,Q)$, $x_0 \sim \mathcal{N}(0,P_0)$, and $w_t \perp v_t$. We also assume that the pairs (A,B) and (A,Q) are controllable, the pair (A,C) is observable, and $R > 0$.

Stabilization is a necessary requirement in any control system, but in addition to that, often performance indices need to be optimized in order to achieve an acceptable behavior of the whole system. A typical choice is the steady state performance in terms of a quadratic cost index as in the linear-quadratic-gaussian (LQG) framework. More formally, in the contest of finite horizon LQG control the cost function is defined as:

$$J = \frac{1}{T}\sum_{t=0}^{T} \mathbb{E}[x_t^\top W x_t + u_t^\top U u_t]$$

while in the infinite horizon LQG control it is given by:

$$\begin{aligned} J &= \lim_{T\to+\infty} \frac{1}{N}\mathbb{E}\left[\sum_{t=0}^{T} x_t^\top W x_t + u_t^\top U u_t\right] \\ &= \lim_{t\to+\infty} \mathbb{E}[x_t^\top W x_t + u_t^\top U u_t] && (3) \end{aligned}$$

where the two limits coincide under customary ergodicity assumptions.We also assume that the pair (A,W) is observable.

Typical choices for the matrices W, U are

$$W = C^T C, \quad U = \rho I \tag{4}$$

that, for $\rho = 0$, yield the so-called *cheap control scenario*, for which $J = \lim_{t\to+\infty} \mathbb{E}[||y_t||^2] - \mathrm{trace}(R)$ that correspond to steady state output minus the measurement noise power. Hence, for ease of notation, in the cheap control scenario we define the cost function as $J := \mathbb{E}\big[\|y_t\|^2\big]$.

The plant output y_t is measured and possibly preprocessed by a causal Coder/Estimator (COD) that sends data a_t across a non-ideal communication channel. At the other end, a causal Decoder/Controller (DEC) processes the data received b_t and computes the control input u_t necessary to stabilize the plant and optimize the performance index J. A pictorial representation is given in top panel of Figure 1.

It is a standard practice to decouple the Coder/Decoder design into two tiers: one associated to the plant (source) and the other associated to the channel, as shown in the bottom panel of Figure 1. The goal of the Plant Coder/Decoder design is to stabilize the closed loop system and possibly to optimize some additional performance index. These blocks in control theory framework correspond to filters, estimators, and controllers. Differently, the goal of the Channel

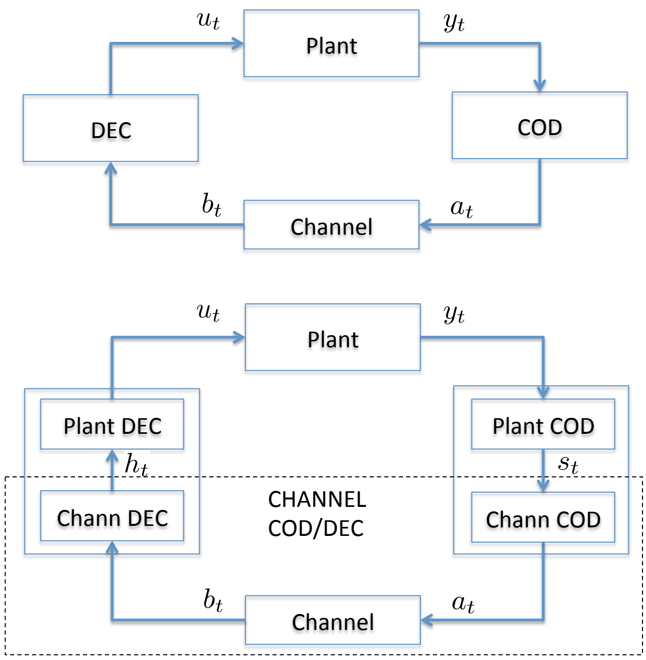

Fig. 1. Scheme of control system across a communication channel: general scheme (top), decoupled scheme (bottom).

Coder/Decoder design is to translate the signal s_t into a signal a_t that is suitable for transmission over the communication channel, in such a way that the signal b_t received from the channel can be decoded into a signal h_t that is as close as possible to the original signal s_t, i.e. $h_t \approx s_t$.[1]

We observe that decoupling coding/decoding into Plant Coding/Decoding and Channel Coding/Decoding may not be the optimal strategy in the context of feedback systems with unstable plants, as suggested by some recent work on anytime capacity and coding/decoding for unstable plants [14]. Nonetheless, we will stick to this approach since greatly simplifies the overall design and is applicable to current communication protocols.

III. Channel Coding/Decoding modeling

As mentioned above, the objective of traditional Channel Coding and Decoding is transfer a (possibly continuous valued) signal s_t across a noisy transmission channel and reconstruct a signal h_t at the receiver, possibly with some delay τ, as closely as possible to original one, i.e. $h_t \approx s_{t-\tau}$.

This is in general obtained via digital communication techniques that require to (i) quantize the real signal s_t into its quantized version s_t^q, (ii) encode it into a string of bits a_t determined by the chosen modulation and coding (iii) transmit this string across the channel and decode it into

[1]Note that, the meaning of Plant and Channel Coding/Decoding considered in this work is different from Source and Channel Coding/Decoding considered in classical Information Theory, where the role of Source Coding is to remove the correlation of the signal y_t to reduce its bit rate, whereas Channel coding adds controlled redundancy to the signal before transmission over the channel to increase its robustness to transmit errors.

another string of bits b_t, (iv) map these strings of bits into a real number h_t.

*Remark 1 (***Scalar output):** For simplicity we shall restrict our attention to systems with scalar output. In fact, if the output is not scalar, the issue of deciding how to encode a vector signal with a finite number of bits is not entirely trivial. Clearly different coding schemes would result in different analog channel models. As such we defer these issues to future work and consider only $m = 1$ from now on in the paper.

The quantization noise $n_t = s_t - s_t^q$ accounts for the distortion due to the quantization of the real-valued signal s_t before transmission. Under the assumption of *fine quantization*, i.e., that the number of quantization levels is sufficiently high, n_t can be effectively modeled as a zero-mean additive random process, with identically distributed uncorrelated samples of power $\sigma_n^2 = \mathbb{E}[n_t^2]$, which is also uncorrelated with s_t. The number of quantization levels also determines the information rate R_q of the quantized signal, which is in turn related to the signal-to-quantization noise ratio (SQNR), $\alpha = \mathbb{E}[s_t^2]/\sigma_n^2$: the larger R_q, the higher the SQNR. The maximum information rate that can be reliably transferred through a communication channel is upper limited by the Shannon capacity R_c of the channel, so it must hold $R_q < R_c$. Accordingly, the SQNR cannot be increased above a certain threshold α^*, which depends on R_c, i.e. $\alpha < \alpha^*(R_c)$. Therefore in our framework, the Shannon Capacity R_c basically determines an upper bound on the possible SQNR α.

The encoding of the quantized signal typically involves two concatenated codes, an inner and an outer code for forward error correction and detection of residual errors (frame check) at the receiver, respectively. As a consequence three scenarios are possible at the receiver: (i) the transmitted string a_t is decoded correctly, i.e. $b_t = a_t$, (ii) the decoded string contains errors that are detected by the outer decoder, which drops the message (*erasure*) (iii) the decoded string contains errors that are not detected by the outer decoder, which accepts a message $b_t \neq a_t$. These events can be modeled via the binary variables $\gamma_t, \nu_t \in \{0, 1\}$, where $\gamma_t = 0$ indicates that an erasure occurred, while $\gamma_t = 1$ denotes that the message was accepted by the receiver, which means that it is either correct or affected by undetected errors. The binary variable ν_t discriminates between these two last events, so that $(\nu_t, \gamma_t) = (1, 1)$ means that an undetected decoding error occurred, and $(\nu_t, \gamma_t) = (0, 1)$ that no error occurred.

The probability of the erasure and undetected error events, denoted by $\epsilon_e := \mathbb{P}[\gamma_t = 0]$ and $\epsilon_0 := \mathbb{P}[\nu_t = 1|\gamma_t = 1]$, respectively, depend on the forward error correction and frame check schemes adopted. In case of undetected errors, the difference between the reconstructed signal h_t and the quantized signal at the transmitter s_t^q, i.e. $m_t = h_t - s_t^q$, would typically be much larger than the quantization noise n_t. For the sake of completeness, we hence included it in the proposed general channel model, though we observe that redundancy codes are usually designed in order to guarantee $\epsilon_0 \ll \epsilon_e$, so that undetected error event is typically negligible in modern communication systems. Finally, the decoding of the string b_t from past strings a_t requires some decoding delay τ, therefore the reconstructed signal h_t is related to the transmitted signal $s_{t-\tau}$.

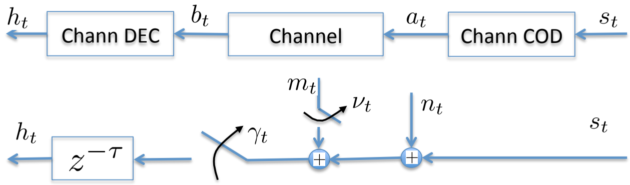

Fig. 2. Equivalent model of Channel COD/DEC using traditional codes

This mathematical modeling of a digital communication channel is summarized in Fig. 2, and it is characterized by the parameters $\epsilon_0, \epsilon_e, \alpha, \tau$. These parameters are clearly dependent, since, for example, reducing the erasure probability ϵ_e requires to increase the delay τ or reduce the SQNR α. A discussion about this dependence is beyond the scope of this work and it is left for future work.

IV. LQG ARCHITECTURE

Based on the channel model described above, which is independent of the control application, our goal is therefore to optimally design the Plant Coding block $\mathcal{F}_t$ and the Plant Decoding block $\mathcal{G}_t$ to minimize the performance cost J, and gain insights in the role played by the channel parameters $(\sigma^2, \epsilon, \tau)$ depicted in Figure 3 in this regard.

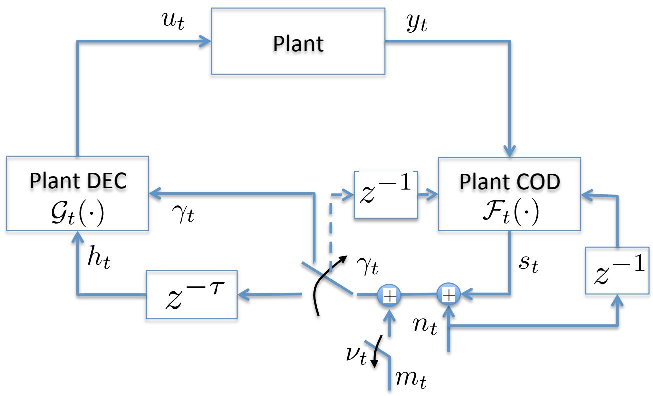

Fig. 3. General scheme of Networked Control System model with implicit channel COD/DEC

The encoder and the decoder can be time-varying and must be causal, i.e. depend only on the past information set. We denote the history of a generic signal f up to time t as $f^t = (f_t, f_{t-1}, \ldots, f_0)$. The information set available to the coder always includes the plant outputs y^t and the past quantization errors n^{t-1}. However, it may also include the past channel erasure γ^{t-1} in case of *perfect channel feedback* from the receiver to the transmitter, which can be practically implemented via a reliable ACK mechanism. Note that in this scenario, if the plant decoder $\mathcal{G}_t$ is deterministic and known to the plant coder $\mathcal{F}_t$, then perfect channel feedback implies that the plant coder can reconstruct the past plant inputs $u^{t-\tau}$, expect in case of undetected errors. The two scenarios (respectively, without and with channel feedback) are summarized in the following equations:

$$\begin{aligned} s_t &= \mathcal{F}_t(y^t, s^{t-1}, n^{t-1}) & (5)\\ s_t &= \mathcal{F}_t(y^t, s^{t-1}, n^{t-1}, \gamma^{t-1}) & (6)\end{aligned}$$

The information set of the decoder includes, besides its past outputs u^{t-1}, also the output from the channel decoder h^t and packet loss sequence γ^t, i.e.

$$u_t = \mathcal{G}_t(h^t, \gamma^t, u^{t-1}) \qquad (7)$$

Note that the decoding error events ν^t are not known to the receiver nor to the transmitter.

In the framework developed above, the objective is to solve the following optimization problem:

$$\begin{aligned} &\min_{\mathcal{F}_t,\mathcal{G}_t} \quad J & (8)\\ &\text{s.t.} \quad \frac{\mathbb{E}[s_t^2]}{\mathbb{E}[n_t^2]} \leq \alpha & (9)\end{aligned}$$

This a formidable optimization problem since it poses only mild conditions on the possible classes of control functions $\mathcal{F}$ and $\mathcal{G}$, which leads to a large design parameter space. Most of the channel models and control architectures studied in the context of NCS can be cast as a special case of the optimization problem (8)-(9).

In general, the channel parameters $(\alpha, \epsilon_e, \tau)$ are assumed to be given. Moreover, they are studied singularly. For example, great attention has been given to lossy communication where only the packet loss parameter ϵ is considered and the SQNR constraint given in Eqn. (9) is neglected [19], [15], [6], [7]. Another area of active research is the SQNR-constrained control that corresponds to the problem where the channel model includes only the quantization noise σ_n^2 and the channel power constrain Eqn. (9) [2], [13], [17], [18], [3]. Only recently, there is an attempt to consider more realistic channel models, for instance by including both packet loss and quantization distortion [22], [8].

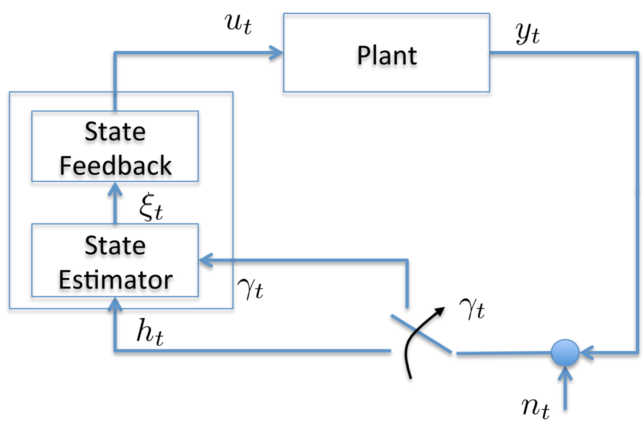

Fig. 4. Special scheme of Networked Control System for scalar output plants

In this work we address the special case of the general optimization problem (8)-(9) shown in Fig. 4. We consider a channel model that includes simultaneously i.i.d packet loss, quantization noise and a limited SQNR α, but assumes no decoding delay and no undetected packet error, i.e.

$$\tau = 0, \quad \mathbb{P}[\nu_t = 1] = 0, \quad \mathbb{P}[\gamma_t = 0] = \epsilon, \quad \sigma^2 = \alpha\,\mathbb{E}[s_t^2] \qquad (10)$$

Moreover, we restrict our design space to the Plant Decoding block and we consider no Plant Coding, i.e.

$$s_t = \mathcal{F}_t(y^t, s^{t-1}) = y_t \qquad (11)$$

Finally, we restrict the Plant Decoder to be obtained with the cascade of a linear state estimator and a state feedback, i.e.,

$$\begin{aligned} \xi_t &= A\xi_{t-1} + Bu_{t-1} + \gamma_t K\big(h_t - C(A\xi_{t-1} + Bu_{t-1})\big) & (12)\\ u_t &= L\xi_t & (13)\\ h_t &= \gamma_t(y_t + n_t) & (14)\end{aligned}$$

Note that the estimator is time-varying since it depends on the sequence γ_t. In fact, if a packet is not received correctly, i.e. $\gamma_t = 0$, then the state estimator updates its state using the model only, while if it is received, i.e. $\gamma_t = 1$, it includes a correction term based on the output innovation similarly to a Kalman filter. This scheme is the same proposed in [15], and does not coincide with the true optimal Kalman filter as in [19]. However, it has the advantage of being computationally simpler and allowing explicit computation of the performance J, as will be shown in the next section.

Finally we will restrict our attention to the *cheap-control setting*, i.e. in the scenario where no penalty is placed on the control input u_t and the objective is to minimize the power of the output plant $\mathbb{E}[||y_t||^2]$.

V. Dynamical equations

We now derive the dynamical equation which governs the state as well as the error evolution for the estimator in equations (12). In order to do so it is convenient to consider the "predictor" $\hat{x}_t$ so that

$$\begin{aligned} \hat{x}_{t+1} &= A\hat{x}_t + Bu_t + \gamma_t G\left[h_t - C\hat{x}_t\right] \\ \xi_t &= \hat{x}_t + \gamma_t K\left[h_t - C\hat{x}_t\right] \qquad (15)\\ u_t &= L\xi_t = L\left[I - \gamma_t KC\right]\hat{x}_t + \gamma_t LKh_t \end{aligned}$$

where $G := AK$. This system has the form of a "Kalman-like" estimator with constant gain K. If the gain K is chosen to be the "optimal" Kalman gain,[2] which we shall denote by K^*, then $\hat{x}_t = \hat{x}_{t|t-1}$ and $\xi_t = \hat{x}(t|t)$, which are respectively the optimal (constant gain) one-step-ahead predictor and estimator of the state x_t.

Consider the error $\tilde{x}_t := x_t - \hat{x}_t$. The "error" equations are therefore:

$$\tilde{x}_{t+1} = x_{t+1} - \hat{x}_{t+1} = Ax_t + Bu_t +$$
$$+ v_t - A\hat{x}_t - Bu_t - \gamma_t G\left[z_t - C\hat{x}_t\right] =$$
$$= A\tilde{x}_t + v_t - \gamma_t G\left[C\tilde{x}_t + w_t + n_t\right] =$$

[2] Within the class of constant gain linear estimators.

$$= (A - \gamma_t GC)\tilde{x}_t + v_t - \gamma_t G(w_t + n_t)$$

Substituting the input given by the controller in the predictor equations:

$$\hat{x}_{t+1} = A\hat{x}_t + Bu_t + \gamma_t G\left[z_t - C\hat{x}_t\right] =$$
$$= A\hat{x}_t + BL\left[I - \gamma_t KC\right]\hat{x}_t + \gamma_t BLKz_t +$$
$$+\gamma_t G\left[z_t - C\hat{x}_t\right]$$
$$= \left[A + BL\right]\hat{x}_t + \gamma_t\left[BLK + G\right]\left[z_t - C\hat{x}_t\right] =$$
$$= \left[A + BL\right]\hat{x}_t + \gamma_t\left[BL + A\right]K\left[C\tilde{x}_t + w_t + n_t\right]$$

The system output is therefore:

$$y_t = Cx_t + w_t = C\left[\tilde{x}_t + \hat{x}_t\right] + w_t$$

It follows that the equation of the feedback loop system are:

$$\begin{bmatrix} \hat{x}_{t+1} \\ \tilde{x}_{t+1} \end{bmatrix} = \begin{bmatrix} (A + BL) & \gamma_t(A + BL)KC \\ 0 & A(I - \gamma_t KC) \end{bmatrix} \begin{bmatrix} \hat{x}_t \\ \tilde{x}_t \end{bmatrix} +$$
$$+ \begin{bmatrix} 0 \\ I \end{bmatrix} v_t + \begin{bmatrix} \gamma_t(A + BL)K \\ -\gamma_t AK \end{bmatrix} \begin{bmatrix} w_t + n_t \end{bmatrix}$$
$$y_t = \begin{bmatrix} C & C \end{bmatrix} \begin{bmatrix} \hat{x}_t \\ \tilde{x}_t \end{bmatrix} + w_t$$

where we use $G = AK$. Defining

$$P := \mathrm{Var}\{[\hat{x}_t^\top, \tilde{x}_t^\top]^\top\} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$$
$$\bar{A}_\gamma := \begin{bmatrix} (A + BL) & \gamma(A + BL)KC \\ 0 & A(I - \gamma KC) \end{bmatrix}$$

and using the fact that $\begin{bmatrix} \hat{x}_t \\ \tilde{x}_t \end{bmatrix}$, v_t and $w_t + n_t$ are pairwise uncorrelated, it follows that:

$$\begin{aligned} P = \quad & (1-\epsilon)\bar{A}_1 P \bar{A}_1^\top + \epsilon \bar{A}_0 P \bar{A}_0^\top + \\ & + \begin{bmatrix} 0 \\ I \end{bmatrix} Q \begin{bmatrix} 0 & I \end{bmatrix} + \\ & + (1-\epsilon)\begin{bmatrix} (A + BL)K \\ -AK \end{bmatrix} [R + N] \begin{bmatrix} (A + BL)K \\ -AK \end{bmatrix}^\top \end{aligned} \tag{16}$$

$$N = \alpha P_y \qquad P_y = \begin{bmatrix} C & C \end{bmatrix} P \begin{bmatrix} C^\top \\ C^\top \end{bmatrix} + R$$

Substituting the expression for P_y in (16) we obtain:

$$\begin{aligned} P = \quad & (1-\epsilon)\bar{A}_1 P \bar{A}_1^\top + \epsilon \bar{A}_0 P \bar{A}_0^\top + \\ & + \begin{bmatrix} 0 \\ I \end{bmatrix} Q \begin{bmatrix} 0 & I \end{bmatrix} + \\ & + (1-\epsilon)(1+\alpha)\begin{bmatrix} (A + BL)K \\ -AK \end{bmatrix} R \begin{bmatrix} (A + BL)K \\ -AK \end{bmatrix}^\top + \\ & + \alpha(1-\epsilon)\bar{\Phi} P \bar{\Phi}^\top \end{aligned} \tag{17}$$

where

$$\bar{\Phi} := \begin{bmatrix} (A + BL)KC & (A + BL)KC \\ -AKC & -AKC \end{bmatrix}$$

For ease of notation we define the operator on the right hand side of (17) as $\mathcal{M}(K, L, P)$, so that (17) can be written in compact form as

$$P = \mathcal{M}(K, L, P)$$

In this work we consider as cost function only the steady state plant output $\mathbb{E}[||y_t||^2]$ in Eqn. (3), i.e. we assume no cost in energy expenditure for the control. This is known as the *cheap-control* scenario in LQG control and it is equivalent of setting $W := C^\top C$ and $\rho = 0$ in Eqn. (3) which takes the form:

$$J \quad = \quad \mathbb{E}\left[x_t^\top C^\top C x_t\right] = \begin{bmatrix} C & C \end{bmatrix} P \begin{bmatrix} C^\top \\ C^\top \end{bmatrix} \tag{18}$$

Hence, the LQG-type optimal control problem can be written as:

$$\begin{aligned} J^* := \min_{K,L} \quad & J \\ s.t. \quad & P = \mathcal{M}(K, L, P) \\ & P \geq 0 \end{aligned} \tag{19}$$

and L^*, K^* will denote the optimal gains.

VI. Solution to the optimal cheap-control scenario

We now derive the solution to the LQG-type optimal control problem (19). The proof technique is borrowed from [4] and goes through the introduction of the Lagrangian

$$\begin{aligned} \mathcal{L}(P, \Lambda, L, K) &:= J + \mathrm{Tr}\{\Lambda\left(P - \mathcal{M}(K, L, P)\right)\} \\ s.t. \quad & P = P^\top \geq 0 \quad \Lambda = \Lambda^\top \geq 0 \end{aligned} \tag{20}$$

Accorning to the matrix maximum principle [1] the necessary conditions for optimality of K^* and L^* are

$$\frac{\partial \mathcal{L}}{\partial P} = 0 \quad \frac{\partial \mathcal{L}}{\partial \Lambda} = 0 \quad \frac{\partial \mathcal{L}}{\partial L} = 0 \quad \frac{\partial \mathcal{L}}{\partial K} = 0 \tag{21}$$

For future reference let us introduce the partition

$$\Lambda := \begin{bmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \end{bmatrix}$$

where all blocks have size $n \times n$. The following proposition summarizes the optimality conditions.

Proposition 1: Consider the LQG-type control problem (19); the optimal gains K^*, L^* can be found solving the necessary conditions (21) for stationarity of the Lagrangian (20) and are given by

$$\begin{aligned} K^* &= P_{22}^* C^\top \Sigma_\alpha^{-1} \\ L^* &= -\left(B^\top \Lambda_{11}^* B\right)^{-1} B^\top \Lambda_{11}^* A \end{aligned} \tag{22}$$

where

$$\Sigma_\alpha := \left(1 + \frac{1}{\alpha}\right)\left(R + CP_{22}^* C^\top\right) + \frac{1}{\alpha} CP_{11}^* C^\top \tag{23}$$

and P_{11}^*, P_{22}^*, Λ_{11}^* and Λ_{22}^* are found solving the following (coupled) Riccati-type equations

$$\begin{array}{rcl} P_{11}^* & = & A_{L^*} P_{11}^* A_{L^*}^\top + \\ & & +(1-\epsilon) A_{L^*} P_{22}^* C^\top \Sigma_\alpha^{-1} C P_{22}^* A_{L^*}^\top \\ \\ P_{22}^* & = & A P_{22}^* A^\top + Q + \\ & & -(1-\epsilon) A P_{22}^* C^\top \Sigma_\alpha^{-1} C P_{22}^* A^\top \\ \\ \Lambda_{11}^* & = & A_{L^*}^\top \Lambda_{11}^* A_{L^*} \\ & & -\frac{1-\epsilon}{\alpha} C^\top (K^*)^\top A_{L^*}^\top \Lambda_{11}^* A_{L^*} K^* C + \\ & & +\frac{1-\epsilon}{\alpha} C^\top (K^*)^\top A^\top \Lambda_{22}^* A K^* C + C^\top C \\ \\ \Lambda_{22}^* & = & \epsilon A^\top \Lambda_{22}^* A - \frac{1-\epsilon}{\alpha} C^\top (K^*)^\top A^\top \Lambda_{22}^* A K^* C + \\ & & (1-\epsilon)(I - K^* C)^\top A^\top \Lambda_{22}^* A (I - K^* C) + \\ & & +(1-\epsilon) A_{L^*}^\top \Lambda_{11}^* A_{L^*} + \\ & & -(1-\epsilon)(I - K^* C)^\top A_{L^*}^\top \Lambda_{11}^* A_{L^*} (I - K^* C) + \\ & & +\frac{1-\epsilon}{\alpha} C^\top (K^*)^\top A_{L^*}^\top \Lambda_{11}^* A_{L^*} K^* C + \\ & & +C^\top C + Q \end{array} \tag{24}$$

where $A_{L^*} := A + BL^*$.

Sketch of the proof: In the interest of space we shall not include a detailed proof (which is straightforward but tedious) and rather summarize the main steps. It can be shown that the necessary conditions (21) are satisfied for P^*, Λ^*, L^*, K^* where

$$P^* := \left[\begin{array}{cc} P_{11}^* & 0 \\ 0 & P_{22} \end{array} \right] \quad \Lambda^* := \left[\begin{array}{cc} \Lambda_{11}^* & \Lambda_{11}^* \\ \Lambda_{11}^* & \Lambda_{22}^* \end{array} \right]$$

and P_{11}^*, P_{22}^*, Λ_{11}^* and Λ_{22}^*, L^* and K^* are defined in (24) and (22). Following the same arguments found in [4] one can show that this is the unique solution, from which the thesis follows.

$\diamond$

*Remark 2 (***Loss of separation principle***):* It is clear that, in general, the optimal values L^* and K^* cannot be found separately and, hence, the separation principle does not hold. In fact it is apparent from (23) and (24) that the (steady state) state error covariance P_{22}^* depends on the state variance P_{11}^*, and hence on the control gain L^*. It is interesting to observe that this dependence vanishes when $\alpha \to \infty$ and, then, the separation principle holds asymptotically, as the quantisation noise power tends to zero and, in turn, the required channel capacity grows to infinity. In fact K^* depends only on P_{22}^* and L^* only on Λ_{11}^* and the equations for Λ_{11}^* and P_{22}^* are decoupled (note that $\Sigma_\infty = R + CP_{22}^* C^\top$). This is coherent with the findings in [20] where the same happens when the control packet arrival probability ν is equal to one (see Fig. 2 and equations (24-29) in [20]).

A. Analysis for B invertible

In order to get more insight on the role of the packet loss probability ϵ and signal to quantization noise ratio α we consider a MISO systems where the input-to-state matrix B is square and invertible, i.e. we have n independent control inputs. Recall also that we have restricted our attention to the scalar output case, so that the matrix C is a row-vector. Under this scenario, the optimal gain L^* in (22) reduces to

$$\begin{array}{rcl} L^* & = & -\left(B^\top \Lambda_{11}^* B\right)^{-1} B^\top \Lambda_{11}^* A \\ & = & -B^{-1} A \end{array} \tag{25}$$

where, as argued in [4], $\Lambda_{11} > 0$ has been used.

For this choice of L^* it is apparent that $P_{11}^* = 0$ so that the steady state variance reduces to

$$P_{11}^* = 0 \tag{26}$$

$$P_{22}^* = A P_{22}^* A^\top + Q + \tag{27}$$

$$-(1-\epsilon) A P_{22}^* C^\top \Sigma_\alpha^{-1} C P_{22}^* A^\top \tag{28}$$

$$\Sigma_\alpha = \left(1 + \frac{1}{\alpha}\right)\left(R + CP_{22}^* C^\top\right) \tag{29}$$

Combining these last two equations we get

$$\begin{array}{rcl} P_{22}^* & = & A P_{22}^* A^\top + Q \\ & & -\eta A P_{22}^* C^\top (C P_{22}^* C^\top + R)^{-1} C P_{22}^* A^\top \\ \eta & := & \frac{1-\epsilon}{1-\frac{1}{\alpha}} \end{array} \tag{30}$$

and the optimal gain

$$K^* = \eta P_{22}^* C^\top (C P_{22}^* C^\top + R)^{-1} \tag{31}$$

Using the results in [16] it is known that when C is rank one, the matrix P_{22}^* in (30) exists and is unique if and only if $\eta > 1 - \frac{1}{\prod_i |\lambda_i^u|^2}$, where $\{\lambda_i^u\}$ represent the unstable eigenvalues of the matrix A. Therefore, from (30), we get

$$\frac{1-\epsilon}{1+\frac{1}{\alpha}} > 1 - \frac{1}{\prod_i |\lambda_i^u|^2} \tag{32}$$

from which we observe that packet erasure probability ϵ, and SQNR α concur in determining the system performance, as briefly discussed below.

B. Discussion and related work

We have seen that the design of the optimal control gain L and the estimator gain K are coupled; the separation principle is recaptured when either the signal to quantization noise ratio α goes to infinity or B is invertible. It is also expected that when B is not invertible the stability condition of Eqn. (32) is likely to provide a condition that is only necessary for stability but not sufficient.

Note also that we recover some of the results available in the literature as special cases. In fact if we let $\alpha \to \infty$, then this is equivalent to consider a channel with infinite capacity and we obtain the same stability condition in the lossy network literature [19], [16]. Alternatively, if we assume no packet loss in the channel, i.e. $\epsilon = 0$, and recalling that $\alpha \le \alpha^*$, then the stability condition can be rewritten as

$$1 - \frac{1}{\prod_i |\lambda_i^u|^2} < \frac{1}{1+\frac{1}{\alpha^*}} = 1 - \frac{1}{1+\alpha^*}$$

that leads to

$$\mathrm{SQNR}^* > \prod_i |\lambda_i^u|^2 - 1$$

which is the same stability condition presented in the context of SNR-limited control system in [2].

Finally, the bound provided by Eqn. (32) will be useful to compare different communication protocols. In fact, by using a corse quantizer, it is possible to reduce the transmission rate R_q, thus allowing more redundant channel coding schemes and consequently a smaller packet loss probability ϵ. However, a coarser quantizer also gives a smaller $\alpha = \mathrm{SQNR}^*$. Conversely, when the channel capacity is limited, increasing α will require an increase of the transmit rate R_q that, in turn, may yield higher packet loss rates. Therefore, α and ϵ are generally coupled and cannot be designed separately. In a scenario where ϵ and α are decoupled, e.g., when packet losses are mainly due to random interference bursts produced by external emitters, a large SQNR will loose the constraint on the erasure probability looses, thus increasing the robustness of the system to packet losses.

VII. Conclusions and future work

We have considered an LQG cheap-control problem under communication constraints. In particular, we proposed a model that accounts for signal quantization, packet erasure, packet error and delay. We have argued in fact that there is a tight connection between these parameters that depends ultimately in channel Shannon Capacity. We then have restricted our attention to a specific control architecture in which the plant outputs are transmitted via a bandlimited channel and then processed through the cascade of a state estimator followed by a linear (state) feedback controller; for ease of exposition we did not consider delays, while both limited rate and packet drops have been included in our analysis. We have derived the optimal solution for a general MISO system under the "cheap control" scenario (i.e. no penalty on the control signal) and showed that the separation principle does not hold in general. When the input-to-state matrix B is invertible the optimal controller has a dead-beat structure and the optimal estimator is a Kalman-like constant gain estimator (which accounts for the packet drop probability). Conditions for stability are derived in terms a modified algebraic Riccati equation and recapture results from the literature as special cases.

Future work will include the analysis of the general LQG problem, i.e. $\rho \neq 0$ for MIMO plants, i.e. $y \in \mathbb{R}^m, m > 1$, and we will explore more sophisticated control schemes with possible compensator before transmission, i.e. $\mathcal{F}_t \neq Identity$.

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